

LIEB–THIRRING BOUNDS FOR COMPLEX JACOBI MATRICES

L. GOLINSKII, S. KUPIN

ABSTRACT. We obtain various versions of classical Lieb–Thirring bounds for one- and multi-dimensional complex Jacobi matrices. Our method is based on Fan–Mirski Lemma and seems to be fairly general.

INTRODUCTION

In a recent interesting paper [5], Frank–Laptev–Lieb–Seiringer obtain Lieb–Thirring bounds for a multidimensional Schrödinger operator $H = -\Delta + V$ with a complex-valued potential. The authors say that they “can also replace $-\Delta$ in H by any operator for which Lieb–Thirring bounds for real-valued potentials hold (but making the appropriate change in the exponent of V on the right side of the inequalities)”. The proposition seems to describe a complex-valued *diagonal perturbation* of a given self-adjoint operator. The method of the paper relies, though, on the special form of the unperturbed self-adjoint operator.

We move somewhat further in this direction. Namely, we prove Lieb–Thirring bounds for a *non-selfadjoint* operator A provided the bounds for its real part $\operatorname{Re} A = (A + A^*)/2$ are available. We neither assume A to be a diagonal perturbation of a self-adjoint operator A_0 , nor we use the specifics of A_0 .

The idea of the proof of the main result is very simple and transparent. First, Lieb–Thirring bounds for complex-valued Jacobi matrices are reduced to the self-adjoint case with the help of an elementary Fan–Mirski Lemma (see [1, Proposition III.5.3]). Then we use results of Hundertmark–Simon [8] for the self-adjoint Jacobi matrices. Since the latter paper contains “small coupling” and “large coupling” bounds, we get pairs of estimates for every case we consider.

More precisely, we are interested in the complex-valued symmetric Jacobi matrices of the form

$$(0.1) \quad J = J(\{a_k\}, \{b_k\}) = \begin{bmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $a_k, b_k \in \mathbb{C}$. We assume J to be a compact perturbation of the free Jacobi matrix $J_0 = J(\{1\}, \{0\})$, or, equivalently, $\lim_{k \rightarrow +\infty} a_k = 1$,

Date: February 5, 2008.

1991 Mathematics Subject Classification. Primary: 47B36, 39A70; Secondary: 34L25.

Key words and phrases. Lieb–Thirring inequalities, one- and multi-dimensional Jacobi matrices, Fan–Mirski Lemma.

$\lim_{k \rightarrow +\infty} b_k = 0$. It is well-known, that in this situation $\sigma_{ess}(J) = [-2, 2]$. The point spectrum of J is denoted by $\sigma_p(J)$; the eigenvalues $\lambda \in \sigma_p(J)$ have finite algebraic (and geometric) multiplicity, and the set of their limit points is on the interval $[-2, 2]$ (see, e.g., [6, Lemma I.5.2]).

Theorem 0.1. *For $p \geq 1$,*

$$\begin{aligned}
 (0.2) \quad & \sum_{\lambda \in \sigma_p(J)} (\operatorname{Re} \lambda - 2)_+^p + \sum_{\lambda \in \sigma_p(J)} (\operatorname{Re} \lambda + 2)_-^p \\
 & \leq c_p \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1|^{p+1/2} \right), \\
 (0.3) \quad & \sum_{\lambda \in \sigma_p(J)} (\operatorname{Re} \lambda - 2)_+^p + \sum_{\lambda \in \sigma_p(J)} (\operatorname{Re} \lambda + 2)_-^p \\
 & \leq 3^{p-1} \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k|^p + 4 |\operatorname{Re} a_k - 1|^p \right),
 \end{aligned}$$

where

$$(0.4) \quad c_p = \frac{1}{2} 3^{p-1/2} \frac{\Gamma(p+1)}{\Gamma(p+3/2)} \frac{\Gamma(2)}{\Gamma(3/2)}.$$

Above, $x_+ = \max\{x, 0\}$, $x_- = -\min\{x, 0\}$ for $x \in \mathbb{R}$, so

$$x = x_+ - x_-, \quad |x| = x_+ + x_-, \quad (-x)_+ = x_-.$$

Note that Theorem 0.1 is a particular case of more general results, Theorems 1.4 and 1.5, of the same spirit.

The paper is organized in the following way. Section 1 contains the proof of the above theorem along with a number of other results for one-dimensional Jacobi matrices. Following the pattern of [5], we also get estimates on single eigenvalues for the complex Jacobi matrices. Similar theorems on multidimensional Jacobi matrices are in Section 2. Actually, the multidimensional results obviously give the estimates of Section 1. Nevertheless, we prefer to state the bounds for the one-dimensional case explicitly.

We also mention that the proofs of the paper go through for general complex Jacobi matrices, not necessarily symmetric ones. Related problems concerning the geometry and location of the discrete spectrum for such matrices are studied in [2, 3, 4].

1. LIEB-THIRRING BOUNDS FOR ONE-DIMENSIONAL JACOBI MATRICES

Let $J = J(\{a_k\}, \{b_k\})$ be a complex Jacobi matrix (0.1), and $\sigma_p(J) = \{\lambda_j\}$ its point spectrum which consists of eigenvalues of finite algebraic multiplicity. For $\alpha \in \mathbb{R}$, we introduce the functions

$$(1.1) \quad f_\alpha^+(\lambda) = (\operatorname{Re} \lambda - 2) + \alpha \operatorname{Im} \lambda, \quad f_\alpha^-(\lambda) = -(\operatorname{Re} \lambda + 2) - \alpha \operatorname{Im} \lambda,$$

and the half-planes

$$(1.2) \quad \Phi_\alpha^+ = \{\lambda : f_\alpha^+(\lambda) > 0\}, \quad \Phi_\alpha^- = \{\lambda : f_\alpha^-(\lambda) > 0\}.$$

It is clear that $\lambda \in \Phi_\alpha^- \Leftrightarrow -\lambda \in \Phi_\alpha^+$. Define also the angles

$$\Psi_\alpha^+ = \Phi_\alpha^+ \cup \Phi_{-\alpha}^+, \quad \Psi_\alpha^- = \Phi_\alpha^- \cup \Phi_{-\alpha}^-.$$

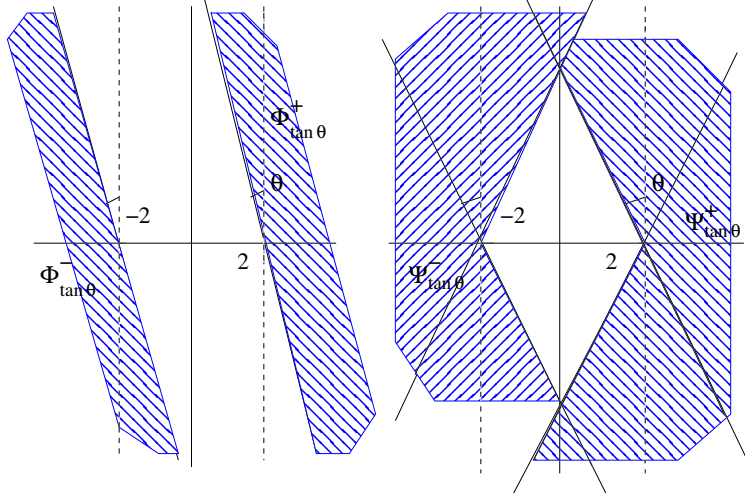


FIGURE 1

For $\alpha = \tan \theta$, $-\pi/2 < \theta < \pi/2$, the regions are represented on Figure 1.

We will be particularly concerned with the parts of the point spectrum $\sigma_p(J)$, lying in Φ_α^\pm and Ψ_α^\pm . We put

$$\sigma_p^\pm(J) = \{\lambda_{\alpha,j}^\pm\} = \sigma_p(J) \cap \Phi_\alpha^\pm,$$

and label the eigenvalues $\{\lambda_{\alpha,j}^\pm\}$ so that

$$(1.3) \quad f_\alpha^\pm(\lambda_{\alpha,1}^\pm) \geq f_\alpha^\pm(\lambda_{\alpha,2}^\pm) \geq \dots > 0, \quad f_\alpha^\pm(\lambda_{\alpha,j}^\pm) \searrow 0.$$

The enumerating takes into account the multiplicities l_j^\pm of $\lambda_{\alpha,j}^\pm$'s, so we have

$$\lambda_1^\pm = \dots = \lambda_{l_1^\pm}^\pm, \quad \lambda_{l_1^\pm+1}^\pm = \dots = \lambda_{l_1^\pm+l_2^\pm}^\pm, \quad \dots$$

For instance, we get for $\alpha = 0$

$$\begin{aligned} f_0^\pm(\lambda) &= \pm(\operatorname{Re} \lambda \mp 2), \quad \Phi_0^\pm = \{\lambda : \pm(\operatorname{Re} \lambda \mp 2) > 0\}, \\ \sigma_p^\pm(J) &= \sigma_p(J) \cap \{\lambda : \pm(\operatorname{Re} \lambda \mp 2) > 0\}. \end{aligned}$$

Clearly, $\lambda \in \sigma_p^-(J) \Leftrightarrow -\lambda \in \sigma_p^+(-J)$.

Furthermore, in the notation $\operatorname{Re} J = (J + J^*)/2$, $\operatorname{Im} J = (J - J^*)/(2i)$, let

$$J_\alpha = \operatorname{Re} J + \alpha \operatorname{Im} J = J(\{\operatorname{Re} a_k + \alpha \operatorname{Im} a_k\}, \{\operatorname{Re} b_k + \alpha \operatorname{Im} b_k\})$$

be a *real self-adjoint* Jacobi matrix, and

$$\sigma_p(J_\alpha) = \sigma_p^-(J_\alpha) \cup \sigma_p^+(J_\alpha) = \{\mu_{\alpha,j}^-\} \cup \{\mu_{\alpha,j}^+\}$$

be the set of its eigenvalues off the essential spectrum $[-2, 2]$, labelled as

$$\mu_{\alpha,1}^- < \mu_{\alpha,2}^- < \dots < -2, \quad \mu_{\alpha,1}^+ > \mu_{\alpha,2}^+ > \dots > 2,$$

and $\lim_{n \rightarrow \infty} \mu_{\alpha,n}^\pm = \pm 2$. In the case when one of the four numbers

$$l^\pm = \sum l_j^\pm = \#\{\lambda_{\alpha,j}^\pm\}, \quad m^\pm = \#\{\mu_j^\pm\}$$

is finite, the natural convention is that $\lambda_{\alpha,j}^{\pm} = \pm 2$, $\mu_{\alpha,k}^{\pm} = \pm 2$ for $j > l^{\pm}$ and $k > m^{\pm}$, respectively. Observe that $\sigma_p^{\pm}(J)$, l^{\pm} , m^{\pm} actually depend on α , but we do not write this dependence to keep the notation reasonably simple.

Lemma 1.1. *We have for $\alpha \in \mathbb{R}$ and $n = 1, 2, \dots$*

$$(1.4) \quad \sum_{j=1}^n ((\operatorname{Re} \lambda_{\alpha,j}^+ - 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^+) \leq \sum_{j=1}^n (\mu_{\alpha,j}^+ - 2),$$

$$(1.5) \quad \sum_{j=1}^n ((\operatorname{Re} \lambda_{\alpha,j}^- + 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^-) \geq \sum_{j=1}^n (\mu_{\alpha,j}^- + 2).$$

Proof. The proof is a combination of the Fan–Mirski Lemma, the classical variational principle for eigenvalues, and elementary properties of a Schur basis for invariant subspaces of an operator [6, Chapter 1], [1]. The argument is essentially finite dimensional, and is implicit in [1, Chapter III].

We will prove the first inequality. The second one, (1.5), is (1.4), applied to $-J$. We denote by $\{\nu_k^+\}$ the eigenvalues of J in Ψ_{α} *without* taking into account their multiplicities. More precisely, $\{\nu_k^+\} = \{\lambda_{\alpha,j}^+\}$ as point sets, and

$$f_{\alpha}^+(\nu_1^+) \geq f_{\alpha}^+(\nu_2^+) \geq \dots > 0, \quad f_{\alpha}^+(\nu_k^+) \searrow 0,$$

but

$$\nu_1^+ = \lambda_{\alpha,1}^+, \quad \nu_k^+ = \lambda_{\alpha, \sum_{j=1}^{k-1} l_j^+ + 1}^+$$

for $k > 1$. In particular, the corresponding root subspaces of J are $H(\nu_k^+) = \operatorname{Ker}(J - \nu_k^+ I)^{l_k^+}$ and $\dim H(\nu_k^+) < +\infty$.

For the transparency of the exposition, we assume that the *geometric multiplicity* of each ν_k^+ is one. In this case $\dim H(\nu_k^+) = l_k^+$. The general situation is treated similarly.

So, let n be an arbitrary positive integer. We distinguish two cases: $n \leq l^+$ and $n > l^+$, $l^+ < +\infty$. For the first one, there exists a unique $k_0 = k_0(n)$ such that

$$n = \sum_{k=1}^{k_0-1} n_k^+ + n', \quad 0 < n' \leq l_{k_0}^+.$$

Put $H'(k_0) = \operatorname{Ker}(J - \nu_{k_0}^+ I)^{n'}$ and consider the direct sum

$$H(n) = H(\nu_1^+) \dot{+} H(\nu_2^+) \dot{+} \dots \dot{+} H(\nu_{k_0-1}^+) \dot{+} H'(k_0).$$

Since the sum is direct and the summands are of finite dimension, we see that the subspace is closed, and $\dim H(n) = n$. It is obvious that $H(n)$ is an invariant subspaces for J , and, by construction, $\sigma_p(J(n)) = \{\nu_k^+\}_{1 \leq k \leq k_0}$, where $J(n) = J|_{H(n)}$.

We now choose the Schur basis $\{x_j\}_{1 \leq j \leq n}$ in $H(n)$. The system $\{x_j\}$ by definition has the following properties:

- it is orthonormal;
- for any $m \leq n$, the linear span of $\{x_j\}_{1 \leq j \leq m}$ is exactly $H(m)$;
- $Jx_j = J(n)x_j = \lambda_{\alpha,j}^+ x_j + \sum_{i=1}^{j-1} a_{j,i} x_i$

with some coefficients $\{a_{j,i}\}$. This means of course that the matrix of $J(n)$ with respect to $\{x_j\}$ is upper-triangular.

Consequently,

$$\lambda_{\alpha,j}^+ = (Jx_j, x_j), \quad \bar{\lambda}_{\alpha,j}^+ = (J^*x_j, x_j),$$

and we have $\operatorname{Re} \lambda_{\alpha,j}^+ = (\operatorname{Re} Jx_j, x_j)$, $\operatorname{Im} \lambda_{\alpha,j}^+ = (\operatorname{Im} Jx_j, x_j)$. So,

$$\operatorname{Re} \lambda_{\alpha,j}^+ + \alpha \operatorname{Im} \lambda_{\alpha,j}^+ = (J_\alpha x_j, x_j),$$

and we continue as

$$(1.6) \quad \sum_{j=1}^n \operatorname{Re} \lambda_{\alpha,j}^+ + \alpha \operatorname{Im} \lambda_{\alpha,j}^+ = \sum_{j=1}^n (J_\alpha x_j, x_j) \leq \sup_{\{y_j\}_{1 \leq j \leq n} \subset H} \sum_{j=1}^n (J_\alpha y_j, y_j),$$

where the supremum is taken over all orthonormal systems $\{y_j\}_{1 \leq j \leq n}$ in H . The operator J_α is self-adjoint and its spectrum is described in the beginning of the current section. The spectral theorem says that

$$J_\alpha = J_\alpha^\perp \oplus (J_\alpha)_p^+, \quad H = H^\perp \oplus H_p^+,$$

where H^\perp, H_p^+ are reducing spectral subspaces associated to $\sigma^\perp(J_\alpha) = \sigma_p^-(J_\alpha) \cup \sigma_{\text{ess}}(J_\alpha)$ and $\sigma_p^+(J_\alpha)$, respectively. Obviously, $((J_\alpha)_p^+ x, x) \geq 2$, $(J_\alpha^\perp y, y) \leq 2$, for $x \in H_p^+$, $y \in H^\perp$, and $\|x\| = \|y\| = 1$. We proceed as

$$\begin{aligned} \text{LHS of (1.6)} &\leq \max_{\{x_j\}_{1 \leq j \leq \min\{n, m_+\}} \subset H_p^+} \sum_j (J_\alpha x_j, x_j) \\ &\quad + \sup_{\{x_j\}_{\min\{n, m_+\}+1 \leq j \leq n} \subset H^\perp} \sum_j (J_\alpha x_j, x_j) \\ &\leq \sum_{j=1}^{\min\{n, m_+\}} \mu_{\alpha,j}^+ + 2(n - m_+)_+, \end{aligned}$$

which is precisely (1.4) under the notational convention made just before the lemma.

When $n > l^+$, we have $\lambda_{\alpha,j}^+ = 2$ and $\mu_{\alpha,j}^+ \geq 2$ for $j > l^+$. So,

$$\sum_{j=1}^n (\operatorname{Re} \lambda_{\alpha,j}^+ + \alpha \operatorname{Im} \lambda_{\alpha,j}^+) = \sum_{j=1}^{l^+} (\operatorname{Re} \lambda_{\alpha,j}^+ + \alpha \operatorname{Im} \lambda_{\alpha,j}^+) + \sum_{j=l^++1}^n 2 \leq \sum_{j=1}^n \mu_{\alpha,j}^+.$$

The proof is complete. \square

Lemma 1.2. For $p \geq 1$ and any $n = 1, 2, \dots$,

$$(1.7) \quad \sum_{j=1}^n ((\operatorname{Re} \lambda_{\alpha,j}^\pm \mp 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^\pm)_\pm^p \leq \sum_{j=1}^n (\mu_{\alpha,j}^\pm \mp 2)_\pm^p.$$

Consequently,

$$\begin{aligned} (1.8) \quad &\sum_{j=1}^n ((\operatorname{Re} \lambda_{\alpha,j}^+ - 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^+)_+^p + \sum_{j=1}^n ((\operatorname{Re} \lambda_{\alpha,j}^- + 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^-)_-^p \\ &\leq \sum_{k=1}^n |\mu_{\alpha,k}^+ - 2|^p + |\mu_{\alpha,k}^- + 2|^p. \end{aligned}$$

Proof. Having (1.4) at hand, the well-known Weyl's lemma [6, Chapter 2], [7] claims that for any non-decreasing convex function g ,

$$\sum_{j=1}^n g((\operatorname{Re} \lambda_{\alpha,j}^+ - 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^+) \leq \sum_{j=1}^n g(\mu_{\alpha,j}^+ - 2),$$

and the function $g(x) = x_+^p$, $p \geq 1$, gives the first inequality in (1.7). We use (1.5) instead of (1.4) to prove the second one. Taking the sum of both bounds (1.7), we get (1.8). \square

Theorem 1.3. *For $\alpha \in \mathbb{R}$ and $p \geq 1$,*

$$(1.9) \quad \sum_{j=1}^{\infty} ((\operatorname{Re} \lambda_{\alpha,j}^+ - 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^+)_+^p + \sum_{j=1}^{\infty} ((\operatorname{Re} \lambda_{\alpha,j}^- + 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^-)_-^p$$

$$\leq c_p \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k + \alpha \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 + \alpha \operatorname{Im} a_k|^{p+1/2} \right),$$

$$(1.10) \quad \sum_{j=1}^{\infty} ((\operatorname{Re} \lambda_{\alpha,j}^+ - 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^+)_+^p + \sum_{j=1}^{\infty} ((\operatorname{Re} \lambda_{\alpha,j}^- + 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^-)_-^p$$

$$\leq 3^{p-1} \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k + \alpha \operatorname{Im} b_k|^p + 4 |\operatorname{Re} a_k - 1 + \alpha \operatorname{Im} a_k|^p \right).$$

Proof. Fix $\alpha \in \mathbb{R}$ and assume that the RHS in (1.9), (1.10) are finite. Then Theorems 2 and 4 from [8] applied to the *self-adjoint* Jacobi matrix J_α , give the desired bounds for $\sum_j |\mu_{\alpha,j}^+ - 2|^p + |\mu_{\alpha,j}^- + 2|^p$. The rest is (1.8) with n going to infinity. \square

It is clear that Theorem 0.1 is precisely (1.9), (1.10) with $\alpha = 0$.

The following result deals with the eigenvalues of J in Ψ_α^\pm .

Theorem 1.4. *For $\alpha \geq 0$ and $p \geq 1$,*

$$(1.11) \quad \sum_{\lambda \in \sigma_p(J)} ((\operatorname{Re} \lambda - 2) + \alpha |\operatorname{Im} \lambda|)_+^p + \sum_{\lambda \in \sigma_p(J)} ((\operatorname{Re} \lambda + 2) - \alpha |\operatorname{Im} \lambda|)_-^p$$

$$\leq c_p \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k + \alpha \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 + \alpha \operatorname{Im} a_k|^{p+1/2} \right.$$

$$\left. + \sum_{k=1}^{\infty} |\operatorname{Re} b_k - \alpha \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 - \alpha \operatorname{Im} a_k|^{p+1/2} \right),$$

$$(1.12) \quad \sum_{\lambda \in \sigma_p(J)} ((\operatorname{Re} \lambda - 2) + \alpha |\operatorname{Im} \lambda|)_+^p + \sum_{\lambda \in \sigma_p(J)} ((\operatorname{Re} \lambda + 2) - \alpha |\operatorname{Im} \lambda|)_-^p$$

$$\leq 3^{p-1} \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k + \alpha \operatorname{Im} b_k|^p + 4 |\operatorname{Re} a_k - 1 + \alpha \operatorname{Im} a_k|^p \right.$$

$$\left. + \sum_{k=1}^{\infty} |\operatorname{Re} b_k - \alpha \operatorname{Im} b_k|^p + 4 |\operatorname{Re} a_k - 1 - \alpha \operatorname{Im} a_k|^p \right).$$

Proof. Let $\alpha \geq 0$. Bound (1.9) implies that

$$\begin{aligned} & \sum_{j: \operatorname{Im} \lambda_{\alpha,j}^+ \geq 0} ((\operatorname{Re} \lambda_{\alpha,j}^+ - 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^+)^p + \sum_{j: \operatorname{Im} \lambda_{\alpha,j}^- \leq 0} ((\operatorname{Re} \lambda_{\alpha,j}^- + 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^-)^p \\ & \leq c_p \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k + \alpha \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 + \alpha \operatorname{Im} a_k|^{p+1/2} \right), \end{aligned}$$

and

$$\begin{aligned} & \sum_{j: \operatorname{Im} \lambda_{-\alpha,j}^+ < 0} ((\operatorname{Re} \lambda_{-\alpha,j}^+ - 2) - \alpha \operatorname{Im} \lambda_{-\alpha,j}^+)^p + \sum_{j: \operatorname{Im} \lambda_{-\alpha,j}^- > 0} ((\operatorname{Re} \lambda_{-\alpha,j}^- + 2) \\ & - \alpha \operatorname{Im} \lambda_{-\alpha,j}^-)^p \leq c_p \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k - \alpha \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 - \alpha \operatorname{Im} a_k|^{p+1/2} \right). \end{aligned}$$

Since

$$\sigma_p(J) \cap \Psi_{\alpha}^{\pm} = \{\lambda_{\alpha,j}^{\pm} : \pm \operatorname{Im} \lambda_{\alpha,j}^{\pm} \geq 0\} \cup \{\lambda_{-\alpha,j}^{\pm} : \pm \operatorname{Im} \lambda_{-\alpha,j}^{\pm} < 0\},$$

we obtain (1.11) adding these two bounds.

Note that transition from α to $-\alpha$ in the above formulae is equivalent to transition from J to J^* . \square

We can refine (1.9) with a bit more precise inequalities

$$\begin{aligned} (1.13) \quad & \sum_j ((\operatorname{Re} \lambda_{\alpha,j}^{\pm} \mp 2) + \alpha \operatorname{Im} \lambda_{\alpha,j}^{\pm})_{\pm}^p \leq c_p \left(\sum_{k=1}^{\infty} (\operatorname{Re} b_k + \alpha \operatorname{Im} b_k)_{\pm}^{p+1/2} \right. \\ & \left. + 2 |\operatorname{Re} a_k - 1 + \alpha \operatorname{Im} a_k|^{p+1/2} \right), \end{aligned}$$

the same applies to (1.10), Theorems 0.1, 1.4 and their multidimensional counterparts, (see the proof of Theorem 1 in [8]).

The “angular” Lieb–Thirring bounds are now an easy consequence of the previous theorem.

Theorem 1.5. *Let $p \geq 1$ and $0 \leq \theta < \pi/2$. Then*

$$\begin{aligned} (1.14) \quad & \sum_{\lambda \in \sigma_p(J) \cap \Psi_{\tan \theta}^+} |\lambda - 2|^p + \sum_{\lambda \in \sigma_p(J) \cap \Psi_{\tan \theta}^-} |\lambda + 2|^p \\ & \leq c_{p,\theta}^1 \left(\sum_{k=1}^{\infty} |b_k|^{p+1/2} + 4 |a_k - 1|^{p+1/2} \right), \end{aligned}$$

$$\begin{aligned} (1.15) \quad & \sum_{\lambda \in \sigma_p(J) \cap \Psi_{\tan \theta}^+} |\lambda - 2|^p + \sum_{\lambda \in \sigma_p(J) \cap \Psi_{\tan \theta}^-} |\lambda + 2|^p \\ & \leq c_{p,\theta}^2 \left(\sum_{k=1}^{\infty} |b_k|^p + 4 |a_k - 1|^p \right), \end{aligned}$$

where

$$c_{p,\theta}^1 = 2^{p/2+5/4} (1 + 2 \tan \theta)^{p+1/2} c_p, \quad c_{p,\theta}^2 = 3^{p-1} 2^{p/2+1} (1 + 2 \tan \theta)^{p/2},$$

and c_p is (0.4).

Proof. We will prove (1.14); the proof of (1.15) is similar.

Given θ , $0 \leq \theta < \pi/2$, we pick θ_1 , $\theta < \theta_1 < \pi/2$, which solves the equation $\tan \theta_1 = 1 + 2 \tan \theta$. Write (1.11) with $\alpha = \tan \theta_1$:

$$\begin{aligned}
 (1.16) \quad & \sum_{\lambda \in \sigma_p(J) \cap \Psi_{\tan \theta}^+} ((\operatorname{Re} \lambda - 2) + \tan \theta_1 |\operatorname{Im} \lambda|)_+^p + \sum_{\lambda \in \sigma_p(J) \cap \Psi_{\tan \theta}^-} ((\operatorname{Re} \lambda + 2) - \tan \theta_1 |\operatorname{Im} \lambda|)^p \\
 & \leq c_p \left(\sum_{k=1}^{\infty} |\operatorname{Re} b_k + \tan \theta_1 \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 + \tan \theta_1 \operatorname{Im} a_k|^{p+1/2} \right. \\
 & \quad \left. + \sum_{k=1}^{\infty} |\operatorname{Re} b_k - \tan \theta_1 \operatorname{Im} b_k|^{p+1/2} + 4 |\operatorname{Re} a_k - 1 - \tan \theta_1 \operatorname{Im} a_k|^{p+1/2} \right).
 \end{aligned}$$

Since $1 \leq \tan \theta_1$ and $a + b \leq \sqrt{2(a^2 + b^2)}$ for $a, b \geq 0$, we have

$$\text{LHS of (1.16)} \leq 2^{1+(p+1/2)/2} \tan^{p+1/2} \theta_1 c_p \left(\sum_{k=1}^{\infty} |b_k|^{p+1/2} + 4 |a_k - 1|^{p+1/2} \right),$$

which is precisely the RHS of (1.14).

Next, in the LHS of (1.16) put $\lambda = x + iy$. If $x - 2 \geq 0$, then $(x - 2) + |y| \tan \theta_1 \geq |\lambda - 2|$. If $x - 2 < 0$, we get

$$\begin{aligned}
 (x - 2) + |y| \tan \theta_1 &= (x - 2) + |y|(1 + 2 \tan \theta) \\
 &= ((x - 2) + |y| \tan \theta) + |y|(1 + \tan \theta) \\
 &\geq |y| + |y| \tan \theta = |y| + (2 - x) \geq |\lambda - 2|,
 \end{aligned}$$

where we used $(x - 2) + |y| \tan \theta \geq 0$ for $\lambda \in \Psi_{\tan \theta}^+$. The second term in the LHS of (1.16) is handled similarly. The theorem is proved. \square

For self-adjoint Jacobi matrices J the bounds for individual eigenvalues $\lambda(J)$ drop out immediately from (1.13)

$$(1.17) \quad (\lambda(J) \mp 2)_{\pm}^p \leq c_p \left(\sum_{k=1}^{\infty} (b_k)_{\pm}^{p+1/2} + 2 |a_k - 1|^{p+1/2} \right),$$

$$(1.18) \quad (\lambda(J) \mp 2)_{\pm}^p \leq 3^{p-1} \left(\sum_{k=1}^{\infty} (b_k)_{\pm}^p + 2 |a_k - 1|^p \right),$$

These estimates, modified appropriately, hold in the non-selfadjoint case as well.

Theorem 1.6. *Let $p \geq 1$, $J = J(\{a_k\}, \{b_k\})$ be a complex Jacobi matrix, and $\lambda(J)$ its eigenvalue. Then*

$$\begin{aligned}
 (1.19) \quad & (\operatorname{Re} \lambda(J) \mp 2)_{\pm}^p \leq c_p \left(\sum_{k=1}^{\infty} (\operatorname{Re} b_k)_{\pm}^{p+1/2} + 2 |\operatorname{Re} a_k - 1|^{p+1/2} \right), \\
 & (\operatorname{Re} \lambda(J) \mp 2)_{\pm}^p \leq 3^{p-1} \left(\sum_{k=1}^{\infty} (\operatorname{Re} b_k)_{\pm}^p + 2 |\operatorname{Re} a_k - 1|^p \right).
 \end{aligned}$$

When $\operatorname{Re} \lambda(J) < -2$ or $\operatorname{Re} \lambda(J) > 2$, we have

(1.20)

$$\begin{aligned} |\lambda(J) \mp 2|^p &\leq 2^{p/2+1/4} c_p \left(\sum_{k=1}^{\infty} |b_k|^{p+1/2} + 2|a_k - 1|^{p+1/2} \right), \\ |\lambda(J) \mp 2|^p &\leq 2^{p/2} 3^{p-1} \left(\sum_{k=1}^{\infty} |b_k|^p + 2|a_k - 1|^p \right). \end{aligned}$$

Finally, when $-2 \leq \operatorname{Re} \lambda(J) \leq 2$, $\lambda(J) \notin [-2, 2]$, we have

(1.21)

$$\begin{aligned} |\lambda(J) \mp 2|^p &\leq c_p (1 + 2 \tan \theta)^{p+1/2} \left(\sum_{k=1}^{\infty} |b_k|^{p+1/2} + 2|a_k - 1|^{p+1/2} \right), \\ |\lambda(J) \mp 2|^p &\leq 3^{p-1} (1 + 2 \tan \theta)^p \left(\sum_{k=1}^{\infty} |b_k|^p + 2|a_k - 1|^p \right), \end{aligned}$$

where θ depends on a particular choice of $\lambda(J)$.

Proof. The bounds in (1.19) and (1.20) are obvious in view of (1.13) (with $\alpha = 0$) and (1.14)–(1.15) (with $\theta = 0$), where a single term in the LHS is taken instead of the whole sum. As far as (1.21) goes, we pick θ in such a way that $\lambda(J) \in \Psi_{\tan \theta}^{\pm}$, and Theorem 1.5 does the rest. \square

2. LIEB–THIRRING ESTIMATES FOR MULTIDIMENSIONAL JACOBI MATRICES

We start recalling the definition of a multidimensional Jacobi matrix acting on $l^2(\mathbb{Z}^\nu)$. Traditionally, the set of unordered pairs $b = (ij)$, $i, j \in \mathbb{Z}^\nu$, $|i - j| = 1$, will be called the set of bonds $B(\mathbb{Z}^\nu)$. For $u = \{u(n)\}_{n \in \mathbb{Z}^\nu} \in l^2(\mathbb{Z}^\nu)$, we define $H = H(\{a_b\}_{b \in B(\mathbb{Z}^\nu)}, \{b(n)\}_{n \in \mathbb{Z}^\nu})$ as

$$\begin{aligned} (2.1) \quad (Hu)(n) &= \sum_{|n-m|=1} a_{(nm)} u(m) + b(n)u(n), \\ (H_0 u)(n) &= \sum_{|m-n|=1} u(m). \end{aligned}$$

where $a_b, b(n) \in \mathbb{C}$. We suppose of course that H is a compact perturbation of H_0 , or, $\lim_{|n| \rightarrow +\infty} a_{(nm)} = 1$ and $\lim_{|n| \rightarrow +\infty} b(n) = 0$. Then obviously $\sigma_{\text{ess}}(H) = [-2\nu, 2\nu]$, and $\sigma_p(H)$ forms a sequence converging to the interval.

One can immediately write down counterparts of *all* results of Section 1 for multidimensional Jacobi matrices. For the sake of brevity, we will illustrate this taking Theorem 0.1 as an example.

The following bounds are obtained in [8, Section 5] for self-adjoint operators H (that is, for $a_b > 0, b(n) \in \mathbb{R}$) and $p \geq 1$:

$$\begin{aligned} \sum_{\lambda \in \sigma_p(H)} (\lambda - 2\nu)_+^p + \sum_{\lambda \in \sigma_p(H)} (\lambda + 2\nu)_-^p \\ \leq 2^\nu (2\nu + 1)^{p+\nu/2-1} L_{p\nu}^{\text{cl}} \left(\sum_n |b(n)|^{p+\nu/2} + 2 \sum_b |a_b - 1|^{p+\nu/2} \right), \end{aligned}$$

$$\begin{aligned}
\sum_{\lambda \in \sigma_p(H)} (\lambda - 2\nu)_+^p &+ \sum_{\lambda \in \sigma_p(H)} (\lambda + 2\nu)_-^p \\
&\leq (2\nu + 1)^{p-1} \left(\sum_n |b(n)|^p + 2 \sum_b |a_b - 1|^p \right).
\end{aligned}$$

where

$$L_{p\nu}^{cl} = 2^{-\nu} \pi^{-\nu/2} \frac{\Gamma(p+1)}{\Gamma(p+\nu/2+1)}.$$

Theorem 2.1. *Let H be a multidimensional complex Jacobi matrix described in (2.1) and $p \geq 1$. Then*

$$\begin{aligned}
\sum_{\lambda \in \sigma_p(H)} (\operatorname{Re} \lambda - 2\nu)_+^p &+ \sum_{\lambda \in \sigma_p(H)} (\operatorname{Re} \lambda + 2\nu)_-^p \\
&\leq 2^\nu (2\nu + 1)^{p+\nu/2-1} L_{p\nu}^{cl} \left(\sum_n |\operatorname{Re} b(n)|^{p+\nu/2} \right. \\
&\quad \left. + 2 \sum_b |\operatorname{Re} a_b - 1|^{p+\nu/2} \right), \\
\sum_{\lambda \in \sigma_p(H)} (\operatorname{Re} \lambda - 2\nu)_+^p &+ \sum_{\lambda \in \sigma_p(H)} (\operatorname{Re} \lambda + 2\nu)_-^p \\
&\leq (2\nu + 1)^{p-1} \left(\sum_n |\operatorname{Re} b(n)|^p + 2 \sum_b |\operatorname{Re} a_b - 1|^p \right).
\end{aligned}$$

REFERENCES

- [1] R. Bhatia, *Matrix analysis*. Springer-Verlag, New York, 1997.
- [2] I. Egorova, L. Golinskii, *Discrete spectrum for complex perturbations of periodic Jacobi matrices*. J. Difference Equ. Appl. **11** (2005), no. 14, 1185–1203.
- [3] I. Egorova, L. Golinskii, *On the location of the discrete spectrum for complex Jacobi matrices*. Proc. Amer. Math. Soc. **133** (2005), no. 12, 3635–3641.
- [4] I. Egorova, L. Golinskii, *On limit sets for the discrete spectrum of complex Jacobi matrices*. Mat. Sb. **196** (2005), no. 6, 43–70; translation in: Sb. Math. **196** (2005), no. 5-6, 817–844.
- [5] R. Frank, A. Laptev, E. Lieb, R. Seiringer, *Lieb-Thirring inequalities for Schrödinger operators with complex-valued potentials*. Lett. Math. Phys. **77** (2006), no. 3, 309–316.
- [6] I. Gohberg, M. Krein, *Introduction to the theory of linear nonselfadjoint operators*. Translations of AMS, vol. 18, AMS, Providence, 1969.
- [7] G. Hardy, J. Littlewood, G. Pólya, *Inequalities*. Cambridge University Press, Cambridge, 1988.
- [8] D. Hundertmark, B. Simon, *Lieb-Thirring inequalities for Jacobi matrices*. J. Approx. Theory **118** (2002), no. 1, 106–130.

MATHEMATICS DIVISION, INSTITUTE FOR LOW TEMPERATURE PHYSICS AND ENGINEERING, 47 LENIN AVE., KHARKOV 61103, UKRAINE

E-mail address: leonid.golinskii@gmail.com

CMI, UNIVERSITÉ DE PROVENCE, 39, RUE JOLIOT-CURIE, 13453 MARSEILLE CEDEX 13, FRANCE

E-mail address: kupin@cmi.univ-mrs.fr